MA 241 : Ordinary Differential Equations (JAN-APR, 2018)

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Problem set 4

If you notice any mistakes in the problems, kindly bring it to my attention

1. Find the Jordan canonical form of the following matrices by finding generalized eigenvalues and generalized eigenvectors. Convert the system $x' = Ax$ to $y' = Ay$ and solve both the systems and draw the phase portraits.

$$i) \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad ii) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad iii) \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$iv) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad v) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad vi) \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

2. Same as (1) for the following 3×3 matrices

$$i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad ii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad iii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$iv) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad v) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \quad vi) \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

3. Do the same for the following 4×4 matrices hold? Do the phase portrait where ever it is possible in $\mathbb{R}^2$ or $\mathbb{R}^3$.

$$i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad ii) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \quad iii) \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad iv) \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

4. i). List all possible upper Jordan canonical form for a 4×4 matrix with a real eigen value λ of multiplicity 4 and give the corresponding deficiency indices in each case.

ii). What is the form of I.V.P in each case?

5. Do the same as in 4 with complex eigenvalues and the solution of the IVP.

6. Using Jordan Decomposition, $A = Pdiag(B_1, B_2, \cdots, B_k)P^{-1}$, if all the eigenvalues of A have negative real part, show that, there are positive constants c and r satisfying the estimate

$$\left|e^{tA}\right| \leq ce^{-rt}$$

for all $t \geq 0$.

7. **Sylvester's Formula:** Let $\lambda_1, \lambda_2, \cdots \lambda_n$ be the distinct eigenvalues of A. Let P_i be the projection operator from $\mathbb{R}^n$ onto the kernel, $\ker(A - \lambda_i I)$ for $i = 1, \cdots n$. Then, show that

$$\begin{array}{rcl} P_1 & +\cdots & +P_n = I \\ \lambda_1 P_1 & +\cdots & +\lambda_n P_n = A \\ \cdots & \cdots & \cdots \\ \lambda_1^{n-1} P_1 & +\cdots & +\lambda_n^{n-1} P_n = A^{n-1} \end{array}$$

The above system can be solved for $P_1, \cdots, P_n$ to get

$$P_i = \prod_{j=1, j\neq i}^{n} \frac{A - \lambda_j I}{\lambda_i - \lambda_j}$$

for $i = 1, \cdots n$. Finally obtain the Sylvester's formula

$$e^{tA} = \sum_{j=1}^{n} e^{t\lambda_j} P_j.$$

8. Using the spectral decomposition of A, decompose $\mathbb{R}^n$ into stable, unstable and center subspaces, respectively, denoted by E^s, E^u and E^c as

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c,$$

where $E^s = span\{u, v : \ a < 0\}$, $E^u = span\{u, v : \ a > 0\}$, $E^c = span\{u, v : \ a = 0\}$. Further, E^s and E^u are invariant with respect to the flow e^{tA}, that is, $e^{tA}E^s \subset E^s$, $e^{tA}E^u \subset E^u$ and $e^{tA}E^c \subset E^c$. Here $u + iv$ represents the generalized eigenvector corresponding to an eigenvalue $\lambda = a + ib$. (note that considering all possible real and complex eigenvlaues).

9. Show that every fundamental matrix $\Psi(t, t_0)$ of a non-autonomous linear system can be written as $\Psi(t, t_0) = \Phi(t, t_0)C$, where C is a constant nonsingular matrix and $\Phi(t, t_0)$ is the transition matrix.

10. When the matrices $A(t)$ and $\displaystyle\int_{t_0}^{t} A(s)\, ds$ commute for all t, show that the transition matrix has the representation

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(s)\, ds\right).$$

11. The above result, in general, is not true if $A(t)$ and $\int_{t_0}^{t} A(s)\, ds$ do not commute. Workout the following matrix to prove it, that is $\mathbf{A}(t) = \begin{pmatrix} 1 & 1+t \\ 0 & t \end{pmatrix}$. Also find the solution to IVP.